2022.9.8

Fact: Any alg. closed field is infinite.

Pf: . .

Unique factorization domain (UFD). Let R be a domain.

- $a \in R$ is irr, if $a \neq 0$, $a \notin R^{\times}$ and $a = bc \Rightarrow b \in R^{\times}$ or $c \in R^{\times}$.
- $R =$ UFD if $\forall a \in R \setminus \{0\}$. $a = u \cdot p_1 \cdots p_n$ $p_i =$ irr, $u \in R^{\times}$.

(uniquely, if $a = u' p_1' \cdots p_m'$ $\exists \sigma \in S_n$ s.t. $p_i \sim p_{\sigma(i)}'$)

Fact: Let R be a UFD with fractional field $K = \mathrm{Frac}(R)$.

(1). $R[x] =$ UFD. ($\Rightarrow R[x_1, \cdots, x_n] =$ UFD)

(2). Let $f \in R[x]$ be a nonconstant polynomial. Then

$$f \text{ irr. in } R[x] \Rightarrow f \text{ irr. in } K[x].$$

(3). $\gcd_{R[x]}(F, G) = 1 \Rightarrow \gcd_{K[x]}(F, G) = 1$.

(4) prime element $\Leftrightarrow$ irreducible element.

Example: PID = principle ideal domain.

(1). PID $\Rightarrow$ UFD

(2). I: nonzero ideal $I = $ max. $\Leftrightarrow$ $I =$ prime   chain of prime ideal

derivative of polynomial.

$$F = \sum a_i X^i \in R[x]. \qquad \frac{\partial F}{\partial X} := F_X := \sum i a_i X^{i-1}$$

$F = \sum_{I} a_I X^I \in R[X_1, \cdots, X_n]$, $\quad I = (i_1, \cdots, i_n)$, $X^I := X_1^{i_1} \cdots X_n^{i_n}$

$$F_{X_k} := \frac{\partial F}{\partial X_k} := \sum_{I} i_k \cdot a_I \cdot X_1^{i_1} \cdots X_k^{i_k - 1} \cdots X_n^{i_n}$$

Fact : (1). $(aF + bG)_X = aF_X + bG_X$, $a, b \in R$

(2). $F_X = 0 \Leftarrow F \in R$

(3). $(FG)_X = F_X \cdot G + F \cdot G_X$ & $(F^n)_X = nF^{n-1} \cdot F_X$

(4). $F(G_1, \cdots, G_n)_X = \sum_{i=1}^{n} F_{X_i}(G_1, \cdots, G_n)\,(G_i)_X$

(5). $(F_{X_i})_{X_j} = (F_{X_j})_{X_i}$

(6). (Euler's thm). F = form of deg m in $R[X_1, \cdots, X_n]$, then

$$mF = \sum_{i=1}^{n} X_i F_{X_i}$$

§1.2. affine space and algebraic sets.

k = field

$$\mathbb{A}^n := \mathbb{A}^n(k) := \underbrace{k \times k \times \cdots \times k}_{n}$$

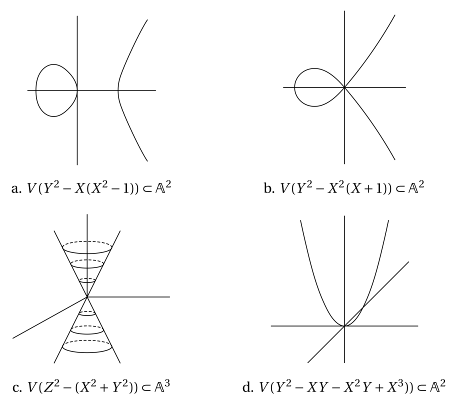
affine n-space

$$V(F) := \{ P = (a_1, \cdots, a_n) \in \mathbb{A}^n \mid F(P) := F(a_1, \cdots, a_n) = 0 \}$$

hypersurface defined by $F \in k[X_1, \cdots, X_n] \setminus k$

- affine plane curve = hypersurface in $\mathbb{A}^2(k)$
- hyperplane = hypersurface in $\mathbb{A}^n(k)$ defined by deg 1 polynomial.

Example. Let $k = \mathbb{R}$



a. $V(Y^2 - X(X^2-1)) \subset \mathbb{A}^2$

b. $V(Y^2 - X^2(X+1)) \subset \mathbb{A}^2$

c. $V(Z^2 - (X^2+Y^2)) \subset \mathbb{A}^3$

d. $V(Y^2 - XY - X^2Y + X^3) \subset \mathbb{A}^2$

affine algebraic set (or, algebraic set) 几何 $\longleftrightarrow$ 代数

$S \subset k[x_1, \cdots, x_n]$

$$V(S) := \{ P \in \mathbb{A}^n \mid F(P) = 0, \ \forall F \in S \}$$

affine algebraic sets

- $V(S) = \bigcap_{F \in S} V(F)$
- $I = (S) \triangleleft k[x_1, \cdots, x_n] \Rightarrow V(S) = V(I)$

  $I \subseteq J \Rightarrow V(I) \supseteq V(J)$

  $V(F_1, \cdots, F_r) := V(\{F_1, \cdots, F_r\})$

Fact: (1). $V(0) = \mathbb{A}^n$, $V(1) = \phi$,

(2). $\bigcap_\alpha V(I_\alpha) = V(\bigcup_\alpha I_\alpha)$

(3). $V(I) \cup V(J) = V(IJ)$

(4). $V(x_1 - a_1, x_2 - a_2, \cdots, x_n - a_n) = \{(a_1, a_2, \cdots, a_n)\}$

Example: classification of alg. subsets in $\mathbb{A}^1(k)$.

Example: 1) $C = \{ r = \sin\theta \} \subseteq \mathbb{A}^2(\mathbb{R})$ ✓

2) $C = \{ (x,y) \mid y = \sin x \} \subseteq \mathbb{A}^2(\mathbb{R})$ ✗

⑧